

# NONLINEAR DIFFERENTIAL EQUATION FOR KOROBV NUMBERS

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**ABSTRACT.** In this paper, we present nonlinear differential equations for the generating functions for the Korobov numbers and for the Frobenius-Euler numbers. As an application, we find an explicit expression for the  $n$ th derivative of  $1/\log(1+t)$ .

**Keywords:** Korobov numbers; Frobenius-Euler numbers

## 1. INTRODUCTION

The *Korobov polynomials*  $K_n(\lambda, x)$  of the first kind are given by

$$\frac{\lambda t}{(1+t)^\lambda - 1} (1+t)^x = \sum_{n \geq 0} K_n(\lambda, x) \frac{t^n}{n!}.$$

For example,

$$\begin{aligned} K_0(\lambda, x) &= 1, \\ K_1(\lambda, x) &= \frac{1}{2}(2x - \lambda + 1), \\ K_2(\lambda, x) &= \frac{1}{12}(6x^2 - 1 + \lambda^2 - 6\lambda x), \\ K_3(\lambda, x) &= \frac{1}{24}(2\lambda x - 2x^2 - \lambda + 2x + 1)(1 - 2x + \lambda). \end{aligned}$$

When  $x = 0$ ,  $K_n(\lambda) = K_n(\lambda, 0)$  are called the *Korobov numbers of the first kind* or just *Korobov numbers*. Since 2002, Korobov polynomials and numbers have been received a lot of attention (see [13, 14]). In particular, these polynomials are used to derive some interpolation formulas of many variables and a discrete analog of the Euler summation formula (see [16]). The *Frobenius-Euler numbers*  $H_n(\mu)$  are defined by the generating function (see [1, 6, 12, 15, 17])

$$\left( \frac{1-\mu}{e^t - \mu} \right) = \sum_{n \geq 0} H_n(\mu) \frac{t^n}{n!}, \quad \mu \neq 1.$$

Recently, the degenerate Bernoulli and Euler polynomials related to Korobov polynomials are studied by several authors ([2–5, 7–12]) and Kim and Kim-Kim derived some interesting identities of Frobenius-Euler polynomials and the Bernoulli polynomials of the second kind arising from nonlinear differential equations (see [4, 7, 8]).

The main goal of this paper is to write a nonlinear differential equation satisfying the generating function  $\frac{\lambda t}{(1+t)^\lambda - 1}$  for Korobov numbers  $K_n(\lambda)$ , and a nonlinear differential equation satisfying the generating function  $\left( \frac{1-\mu}{e^t - \mu} \right)$  for Frobenius-Euler numbers  $H_n(\mu)$ , see next sections. Also, we

present in each case some applications for our nonlinear differential equations. For instance, we find an explicit expression for the  $n$ th derivative of  $1/\log(1+t)$ , see Corollary 3.

## 2. KOROBV NUMBERS

Put  $F = F(t) = F(t; \lambda) = \frac{1}{(1+t)^\lambda - 1}$  ( $\lambda \neq 0$ ). By differentiating respect to  $t$ , we have

$$(1) \quad F^{(1)} = \frac{-1}{((1+t)^\lambda - 1)^2} \cdot \frac{\lambda(1+t)^\lambda}{1+t} = \frac{-\lambda}{1+t} \cdot \frac{(1+t)^\lambda - 1 + 1}{((1+t)^\lambda - 1)^2} = \frac{-\lambda}{1+t} (F + F^2).$$

Now, we let

$$F^{(N)} = \frac{(-1)^N \lambda}{(1+t)^N} \sum_{i=1}^{N+1} a_{i-1}(N; \lambda) F^i = \frac{(-1)^N \lambda}{(1+t)^N} \sum_{i=1}^{N+1} a_{i-1}(N) F^i,$$

for all  $N \geq 0$ , and  $a_i(N) = 0$  for all  $i \geq N+1$ . Note that  $F = F^{(0)} = \lambda a_0(0)F$ , which implies that  $a_0(0) = \frac{1}{\lambda}$ . Also, by (1), we have  $a_0(1) = a_1(1) = 1$ . For  $N+1$ , we have

$$\begin{aligned} F^{(N+1)} &= \frac{d}{dt} \left( \frac{(-1)^N \lambda}{(1+t)^N} \sum_{i=1}^{N+1} a_{i-1}(N) F^i \right) \\ &= \frac{(-1)^{N+1} \lambda N}{(1+t)^{N+1}} \sum_{i=1}^{N+1} a_{i-1}(N) F^i + \frac{(-1)^N \lambda}{(1+t)^N} \sum_{i=1}^{N+1} i a_{i-1}(N) F^{i-1} F^{(1)}, \end{aligned}$$

which, by (1), gives

$$\begin{aligned} F^{(N+1)} &= \frac{(-1)^{N+1} \lambda N}{(1+t)^{N+1}} \sum_{i=1}^{N+1} a_{i-1}(N) F^i + \frac{(-1)^N \lambda}{(1+t)^N} \sum_{i=1}^{N+1} i a_{i-1}(N) F^{i-1} \frac{-\lambda(F + F^2)}{1+t} \\ &= \frac{(-1)^{N+1} \lambda}{(1+t)^{N+1}} \left( \sum_{i=1}^{N+1} N a_{i-1}(N) F^i + \sum_{i=1}^{N+1} \lambda i a_{i-1}(N) F^i + \sum_{i=2}^{N+2} \lambda(i-1) a_{i-2}(N) F^i \right) \end{aligned}$$

By assumption,  $F^{(N+1)} = \frac{(-1)^{N+1} \lambda}{(1+t)^{N+1}} \sum_{i=1}^{N+2} a_{i-1}(N+1) F^i$ , and, by comparing the coefficients of  $F^i$  on both sides, we obtain the following recurrence relation

$$(2) \quad a_{i-1}(N+1) = (N + i\lambda) a_{i-1}(N) + \lambda(i-1) a_{i-2}(N), \quad i = 2, \dots, N+1$$

with  $a_j(N+1) = 0$  whenever  $j \geq N+2$ ,

$$(3) \quad a_0(N+1) = (N + \lambda) a_0(N), \quad a_{N+1}(N+1) = \lambda(N+1) a_N(N).$$

Recalling that  $a_0(0) = \frac{1}{\lambda}$  and  $a_0(1) = a_1(1) = 1$ , by induction on  $N$ , we have

$$\begin{aligned} a_0(N+1) &= (N + \lambda)(N-1 + \lambda) \cdots (1 + \lambda) = (N + \lambda)_N, \\ a_{N+1}(N+1) &= a_1(1) \prod_{j=2}^{N+1} (j\lambda) = \lambda^N (N+1)!. \end{aligned}$$

In next lemma, we treat the general case.

**Lemma 1.** *The coefficients  $a_j(N)$ ,  $j = 1, 2, \dots, N$ , satisfy*

$$a_j(N) = j\lambda \sum_{i=0}^{N-j} (N + (j+1)\lambda - 1)_i a_{j-1}(N - i - 1).$$

*Proof.* By (2), we have

$$\begin{aligned} a_j(N+1) &= j\lambda a_{j-1}(N) + (N + (j+1)\lambda) a_j(N) \\ &= j\lambda a_{j-1}(N) + j\lambda(N + (j+1)\lambda) a_{j-1}(N-1) \\ &\quad + (N + (j+1)\lambda)(N + (j+1)\lambda - 1) a_j(N-1). \end{aligned}$$

By induction and the initial condition  $a_j(j) = \lambda^{j-1}j!$ , we obtain

$$a_j(N+1) = j\lambda \sum_{i=0}^{N+1-j} (N + (j+1)\lambda)_i a_{j-1}(N-i),$$

as required.  $\square$

By Lemma 1, we can state the following result.

**Theorem 2.** *The function  $F = F(t) = \frac{1}{(1+t)^{\lambda-1}}$  with  $\lambda \neq 0$  satisfies the nonlinear differential equation*

$$F^{(N)} = \frac{(-1)^N \lambda}{(1+t)^N} \sum_{i=1}^{N+1} a_{i-1}(N) F^i,$$

where  $a_0(N) = (N + \lambda - 1)_{N-1}$  with  $a_0(0) = \frac{1}{\lambda}$ , and

$$a_j(N) = j\lambda \sum_{i=0}^{N-j} (N + (j+1)\lambda - 1)_i a_{j-1}(N - i - 1),$$

for  $j = 1, 2, \dots, N$ .

As an application for Theorem 2, let us consider the limit  $\lim_{\lambda \rightarrow 0} \lambda F(t)$ . By the fact that  $\lim_{\lambda \rightarrow 0} \lambda F(t) = \frac{1}{\log(1+t)}$ , we obtain

$$\frac{d^N}{dt^N} \frac{1}{\log(1+t)} = \frac{(-1)^N}{(1+t)^N} \sum_{i=2}^{N+1} \lim_{\lambda \rightarrow 0} \lambda^{2-i} a_{i-1}(N; \lambda) \frac{1}{\log^i(1+t)}.$$

Combining with the previous result in [4], we obtain

$$\lim_{\lambda \rightarrow 0} \lambda^{2-i} a_{i-1}(N; \lambda) = (i-1)!(N-1)! H_{N-1, i-2},$$

where  $2 \leq i \leq N+1$  and  $H_{N,j}$  is given by

$$H_{N,j} = \begin{cases} 1, & j = 0, \\ H_N = \sum_{i=1}^N \frac{1}{i}, & j = 1, \\ \sum_{i=1}^N \frac{H_{i-1, j-1}}{i}, & 2 \leq j \leq N, \end{cases}$$

with  $H_{0, j-1} = 0$  when  $j \geq 2$ . Hence, by Theorem 2, we can state the following corollary.

**Corollary 3.** *For all  $N \geq 1$ ,*

$$\frac{d^N}{dt^N} \frac{1}{\log(1+t)} = \frac{(-1)^N (N-1)!}{(1+t)^N} \sum_{i=2}^{N+1} \frac{(i-1)! H_{N-1,i-2}}{\log^i(1+t)}.$$

For example,

$$\begin{aligned} \frac{d}{dt} \frac{1}{\log(1+t)} &= \frac{-1}{1+t} \frac{1}{\log^2(1+t)}, \\ \frac{d^2}{dt^2} \frac{1}{\log(1+t)} &= \frac{1}{(1+t)^2} \left( \frac{1}{\log^2(1+t)} + \frac{2}{\log^3(1+t)} \right), \\ \frac{d^3}{dt^3} \frac{1}{\log(1+t)} &= \frac{-1}{(1+t)^3} \left( \frac{2}{\log^2(1+t)} + \frac{6}{\log^3(1+t)} + \frac{6}{\log^4(1+t)} \right). \end{aligned}$$

As another application, let us consider the generating function for the Korobov numbers

$$\frac{\lambda t}{(1+t)^\lambda - 1} = \sum_{n \geq 0} K_n(\lambda) \frac{t^n}{n!}.$$

More generally, the Korobov numbers of order  $m$  are defined via the following generating function

$$\left( \frac{\lambda t}{(1+t)^\lambda - 1} \right)^m = \sum_{n \geq 0} K_n^{(m)}(\lambda) \frac{t^n}{n!}.$$

**Theorem 4.** *For all  $N \geq 1$ ,*

$$\begin{aligned} \sum_{i=0}^{\min(n,N)} \lambda^{i-N+1} (n)_i a_{N-i}(N) K_{n-i}^{(N+1-i)}(\lambda) \\ = \begin{cases} N!(N)_n, & 0 \leq n \leq N, \\ (-1)^N \sum_{\ell=0}^{n-N-1} \binom{N}{\ell} \frac{K_{n-\ell}(\lambda)}{n-\ell} (n)_{N+1+\ell}, & n \geq N+1. \end{cases} \end{aligned}$$

*Proof.* Note that

$$F = \frac{1}{(1+t)^\lambda - 1} = \frac{1}{\lambda} \left( \frac{1}{t} + \sum_{n \geq 0} K_{n+1}(\lambda) \frac{t^n}{(n+1)!} \right).$$

Thus,

$$F^{(N)} = \frac{1}{\lambda} \left( (-1)^N N! t^{-N-1} + \sum_{n \geq N} K_{n+1}(\lambda) (n)_N \frac{t^{n-N}}{(n+1)!} \right),$$

which implies

$$t^{N+1} F^{(N)} = \frac{1}{\lambda} \left( (-1)^N N! + \sum_{n \geq N} K_{n+1}(\lambda) (n)_N \frac{t^{n+1}}{(n+1)!} \right).$$

Multiplying both sides by  $(1+t)^N$ , we obtain

$$\begin{aligned}
 (1+t)^N t^{N+1} F^{(N)} &= \frac{1}{\lambda} \left( (-1)^N N! \sum_{n=0}^N (N)_n \frac{t^n}{n!} + \sum_{\ell=0}^N (N)_\ell \frac{t^\ell}{\ell!} \sum_{n \geq N} K_{n+1}(\lambda) (n)_N \frac{t^{n+1}}{(n+1)!} \right) \\
 (4) \quad &= \frac{1}{\lambda} \left( (-1)^N N! \sum_{n=0}^N (N)_n \frac{t^n}{n!} + \sum_{n \geq N+1} \sum_{\ell=0}^{n-N-1} \binom{N}{\ell} \frac{K_{n-\ell}(\lambda)}{n-\ell} (n)_{N+1+\ell} \frac{t^n}{n!} \right).
 \end{aligned}$$

On the other hand, by Theorem 2, we have

$$\begin{aligned}
 (1+t)^N t^{N+1} F^{(N)} &= (-1)^N \lambda \sum_{i=1}^{N+1} a_{i-1}(N) \frac{t^{N+1}}{((1+t)^\lambda - 1)^i} \\
 &= (-1)^N \lambda \sum_{i=1}^{N+1} \lambda^{-i} a_{i-1}(N) \left( \frac{\lambda t}{(1+t)^\lambda - 1} \right)^i t^{N+1-i} \\
 &= (-1)^N \lambda \sum_{i=0}^N \lambda^{i-N-1} a_{N-i}(N) \left( \frac{\lambda t}{(1+t)^\lambda - 1} \right)^{N+1-i} t^i.
 \end{aligned}$$

Thus by the generating function for the Korobov numbers. we obtain

$$\begin{aligned}
 (1+t)^N t^{N+1} F^{(N)} &= (-1)^N \lambda \sum_{i=0}^N \left( \lambda^{i-N-1} a_{N-i}(N) t^i \sum_{m \geq 0} K_m^{(N+1-i)}(\lambda) \frac{t^m}{m!} \right) \\
 &= (-1)^N \sum_{n \geq 0} \left( \sum_{i=0}^{\min(n, N)} \lambda^{i-N} (n)_i a_{N-i}(N) K_{n-i}^{(N+1-i)}(\lambda) \right) \frac{t^n}{n!}.
 \end{aligned}$$

By combining this equation with (4), we complete the proof.  $\square$

### 3. FROBENIUS-EULER NUMBERS

Set  $F = F(t) = F(t; \lambda, \mu) = \frac{1}{(1+\lambda t)^{\frac{1}{\lambda} - \mu}}$  ( $\lambda \neq 0$ ). By differentiating respect to  $t$ , we have

$$(5) \quad F^{(1)} = \frac{-1}{((1+\lambda t)^{\frac{1}{\lambda} - \mu})^2} \cdot \frac{(1+\lambda t)^{\frac{1}{\lambda}}}{1+\lambda t} = \frac{-1}{1+\lambda t} \cdot \frac{(1+\lambda t)^{\frac{1}{\lambda} - \mu} - \mu + \mu}{((1+\lambda t)^{\frac{1}{\lambda} - \mu})^2} = \frac{-1}{1+\lambda t} (F + \mu F^2).$$

Now, we let

$$F^{(N)} = \frac{(-1)^N}{(1+\lambda t)^N} \sum_{i=1}^{N+1} b_{i-1}(N; \lambda, \mu) F^i = \frac{(-1)^N}{(1+\lambda t)^N} \sum_{i=1}^{N+1} b_{i-1}(N) F^i,$$

for all  $N \geq 0$ , and  $a_i(N) = 0$  for all  $i \geq N + 1$ . Note that  $F = F^{(0)} = b_0(0)F$ , which implies that  $b_0(0) = 1$ . Also, by (5), we have  $b_0(1) = 1$  and  $b_1(1) = \mu$ . For  $N + 1$ , we have

$$\begin{aligned} F^{(N+1)} &= \frac{d}{dt} \left( \frac{(-1)^N}{(1+\lambda t)^N} \sum_{i=1}^{N+1} b_{i-1}(N) F^i \right) \\ &= \frac{(-1)^{N+1} \lambda N}{(1+\lambda t)^{N+1}} \sum_{i=1}^{N+1} b_{i-1}(N) F^i + \frac{(-1)^N}{(1+\lambda t)^N} \sum_{i=1}^{N+1} i b_{i-1}(N) F^{i-1} F^{(1)}, \end{aligned}$$

which, by (5), gives

$$\begin{aligned} F^{(N+1)} &= \frac{(-1)^{N+1} \lambda N}{(1+\lambda t)^{N+1}} \sum_{i=1}^{N+1} b_{i-1}(N) F^i + \frac{(-1)^N \lambda}{(1+\lambda t)^N} \sum_{i=1}^{N+1} i b_{i-1}(N) F^{i-1} \frac{-(F + \mu F^2)}{1 + \lambda t} \\ &= \frac{(-1)^{N+1}}{(1+\lambda t)^{N+1}} \left( \sum_{i=1}^{N+1} (N\lambda + i) b_{i-1}(N) F^i + \sum_{i=2}^{N+2} (i-1) \mu b_{i-2}(N) F^i \right). \end{aligned}$$

By assumption,  $F^{(N+1)} = \frac{(-1)^{N+1}}{(1+\lambda t)^{N+1}} \sum_{i=1}^{N+2} b_{i-1}(N+1) F^i$ , and, by comparing the coefficients of  $F^i$  on both sides, we obtain the following recurrence relation

$$(6) \quad b_{i-1}(N+1) = (N\lambda + i) b_{i-1}(N) + \mu(i-1) b_{i-2}(N), \quad i = 2, \dots, N+1$$

with  $b_j(N+1) = 0$  whenever  $j \geq N+2$ ,

$$(7) \quad b_0(N+1) = (N\lambda + 1) b_0(N), \quad b_{N+1}(N+1) = \mu(N+1) b_N(N).$$

Recalling that  $b_0(0) = 1$ ,  $b_0(1) = 1$  and  $b_1(1) = \mu$ , by induction on  $N$ , we have

$$b_0(N+1) = (N\lambda + 1)((N-1)\lambda + 1) \cdots (\lambda + 1) = (N\lambda + 1| \lambda)_N,$$

$$b_{N+1}(N+1) = \mu^{N+1} (N+1)!,$$

where  $(x | \lambda)_n = x(x-\lambda) \cdots (x-(n-1)\lambda)$ .

In next lemma, we treat the general case.

**Lemma 5.** *The coefficients  $b_j(N)$ ,  $j = 1, 2, \dots, N$ , satisfy*

$$b_j(N) = j\mu \sum_{i=0}^{N-j} ((N-1)\lambda + j + 1 | \lambda)_i b_{j-1}(N-i-1).$$

*Proof.* By (6), we have

$$\begin{aligned} b_j(N+1) &= j\mu b_{j-1}(N) + (N\lambda + j + 1) b_j(N) \\ &= j\mu b_{j-1}(N) + j\mu(N\lambda + j + 1) b_{j-1}(N-1) \\ &\quad + (N\lambda + j + 1)((N-1)\lambda + j + 1) b_j(N-1). \end{aligned}$$

By induction and the initial condition  $b_j(j) = \mu^j j!$ , we obtain

$$b_j(N+1) = j\mu \sum_{i=0}^{N+1-j} (N\lambda + j + 1 | \lambda)_i b_{j-1}(N-i),$$

as required. □

By Lemma 5, we can state the following result.

**Theorem 6.** *The function  $F = F(t) = \frac{1}{(1+\lambda t)^{\frac{1}{\lambda}-\mu}}$  with  $\lambda \neq 0$  satisfies the nonlinear differential equation*

$$F^{(N)} = \frac{(-1)^N}{(1+\lambda t)^N} \sum_{i=1}^{N+1} b_{i-1}(N) F^i,$$

where  $b_0(N) = ((N-1)\lambda + 1|\lambda)_{N-1} (N \geq 1)$  with  $b_0(0) = 1$ , and

$$b_j(N) = j\mu \sum_{i=0}^{N-j} ((N-1)\lambda + j+1|\lambda)_i b_{j-1}(N-i-1),$$

for  $j = 1, 2, \dots, N$ .

By considering the proof of Theorem 6 and taking  $\lambda \rightarrow 0$ , we obtain the following result.

**Theorem 7.** *Let  $F = F(t) = \frac{1}{e^t - \mu}$ . Then*

$$F^{(N)} = (-1)^N \sum_{i=1}^{N+1} b_{i-1}(N; \mu) F^i,$$

where  $b_0(N; \mu) = b_0(N-1; \mu)$ ,  $b_N(N; \mu) = \mu N b_{N-1}(N-1; \mu)$ , and  $b_{i-1}(N; \mu) = i b_{i-1}(N-1; \mu) + \mu(i-1) b_{i-2}(N-1; \mu)$ , for  $2 \leq i \leq N$  with  $b_0(0; \mu) = b_0(1; \mu) = 1$  and  $b_1(1; \mu) = \mu$ .

By the recurrence relation of  $b_i(N; \mu)$ , see Theorem 7, and by induction on  $N$ , we obtain  $b_0(N; \mu) = 1$  and  $b_N(N; \mu) = \mu^N N!$ , for all  $N \geq 0$ .

Along the lines of the proof of Lemma 5 and taking  $\lambda \rightarrow 0$ , we obtain that the coefficients  $b_j(N; \mu)$ ,  $j = 1, 2, \dots, N$ , satisfy

$$(8) \quad b_j(N; \mu) = j\mu \sum_{i=0}^{N-j} (j+1)^i b_{j-1}(N-i-1; \mu) \text{ with } b_0(N; \mu) = 1.$$

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